More on determinants

Recall Cofactor expansion (3×3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Pick any row/column

- Row 2 : $|\underline{A}| = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$

$$= a_{21} (-1)^{2+1} M_{21} + a_{22} (-1)^{2+2} M_{22} + a_{23} (-1)^{2+3} M_{23}$$

Properties : $|\underline{A}|$ can be negative! (orientation flip)

- $|\underline{A}^T| = |\underline{A}|$ (transpose : rows $\longleftrightarrow$ cols)
- $|\underline{A}| \neq 0 \iff \underline{A}^{-1}$ defined
- If $|\underline{A}| \neq 0$, then $|\underline{A}^{-1}| = \dfrac{1}{|\underline{A}|}$.

(upper) triangular matrix

(entries below main diagonal $= 0$)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

(lower) triangular

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

★ det([triangular matrix])

= (product of diagonal entries)

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24$

Now, can we use row/col ops to make 0's in matrix or make triangular to get $|\underline{A}|$?

Swap : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{(R_1 \leftrightarrow R_2)} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

det = −2   det = +2 = −(−2)

(b/c Cofactor signs $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$)

$$\boxed{\text{swapping two rows (or two columns) flips sign of } \det(\cdots)}$$

Note column ops work similarly b/c:

transpose swaps rows ⟷ columns

$\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{T} \underline{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

(det = −2)   (det = −2)

$(R_1 \leftrightarrow R_2) \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $(C_1 \leftrightarrow C_2) \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

(det = 2)   (det = 2)

Scale : $\boxed{\text{multiplying one row/column by } k \text{ causes det. to get multiplied by } k.}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (det = −2)

$(R_2 \to -3R_2) \begin{bmatrix} 1 & 2 \\ -9 & -12 \end{bmatrix}$ (det = (−3)(−2) = 6)

$\boxed{\text{adding: Ops } (R_i \to R_i + kR_j) \text{ and } (C_i \to C_i + kC_j) \text{ do NOT affect determinant!}}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad (\det = -2)$$

$$(R_2 \to R_2 - 3R_1) \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \quad \text{triangular} \Rightarrow (\det = 1 \cdot (-2) = -2)$$

Ex

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -3 & 2 \\ 5 & 1 & 9 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{pmatrix} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + 3R_3 \end{pmatrix} \begin{bmatrix} 12 & 0 & 21 \\ 18 & 0 & 29 \\ 5 & 1 & 9 \end{bmatrix} \quad \text{no effect on determinant}$$

column 2

$$\det = -0(\cdots) + 0(\cdots) - 1 \begin{vmatrix} 12 & 21 \\ 18 & 29 \end{vmatrix}$$

$$-1\big((360-12) - (360+18)\big)$$

$$-1(-30) = \boxed{30}$$

Ex

$$A = \begin{bmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & 5 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -6 & 3 \end{bmatrix} \quad \text{nicer \#'s.}$$

NET effect

$$(R_1 \leftrightarrow R_3) \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 5 & 4 \\ 2 & 4 & -2 & 6 \\ 0 & 2 & -6 & 3 \end{bmatrix} \qquad \times(-1)$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & -6 & -2 \\ 0 & 2 & -6 & 3 \end{bmatrix} \qquad \times(-1)$$

$$\left(\begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_4 \end{array}\right) \quad \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & -12 & 3 \end{bmatrix} \quad \times(-1)$$

$$(R_4 \to R_4 - R_3) \quad \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \times(-1)$$

triangular $\Rightarrow$ $\det = -60$. Then $\det(\underline{A}) = (-60)(-1)$

$= \boxed{60}$

Adjugate matrix (for 3×3 — works for $4\times4, \dots$)

$$\operatorname{adj}(\underline{A}) \overset{\text{DEF}}{=} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$\leftarrow$ cofactors

Inverse matrix thm: If $\det(\underline{A}) \neq 0$, then

$$\boxed{\underline{A}^{-1} = \frac{\operatorname{adj}(\underline{A})}{\det(\underline{A})} = \frac{1}{|\underline{A}|} \cdot \operatorname{adj}(\underline{A})}$$

Why? Ex: $(1,1)$ entry of $\underline{A}\operatorname{adj}(\underline{A})$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} C_{11} & \cdot & \cdot \\ C_{12} & \cdot & \cdot \\ C_{13} & \cdot & \cdot \end{bmatrix}$$

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = |\underline{A}|$$

so $\underline{A} \cdot \dfrac{\operatorname{adj}(\underline{A})}{|\underline{A}|}$ makes 1's and 0's $\longrightarrow$ $\underline{I_n}$